To J. J. Sylvester Esqre with the Author's
kind regards and [illegible]

1026

HINTS ON PORISMS,

IN A LETTER TO

T. S. DAVIES, Esq. F.R.S. F.S.A. &c.

WITH

A SCHOLIUM NOT CONTAINED IN THE LETTER.

BEING

A SEQUEL TO THE TWO TRACTS ON 'IMAGINARY QUANTITIES,'

PUBLISHED IN 1817 AND 1818,

AS A PARTIAL DEVELOPMENT OF VIEWS THEREIN NOTICED.

BEING Nº III—OF ORIGINAL TRACTS.

(77)

BY

BENJAMIN GOMPERTZ, F.R.S. F.R.A.S. &c.

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To T. S. DAVIES, Esq.

PROFESSOR OF MATHEMATICS, F.R.S., &c.

My Dear Sir,

Your having expressed, in reference to my Tracts, published many years back, on 'Imaginary Quantities,' an inclination to be informed whether I had any ideas on the subject of Porisms, which were not to be found in that publication, I may state, that I feel flattered by the inquiry. So many years have passed since I have been able seriously to attend to mathematical subjects, that I have felt some difficulty in recurring to this speculation; and, consequently, in lieu of giving a direct reply to your inquiry, I consider it preferable, with regard to my own facilities and probably with regard also to your satisfaction, to give some short account of my present views, in endeavouring to "call spirits from the vasty deep," in reference to their development, touching this branch of research; believing that, in so doing, I shall, in a great measure, comprehend views I entertained, but never worked out, though hinted at in the Tracts to which

you allude; because the two Tracts, which were intended to be the commencement of a series of others, which I thought might contain new subjects of interest, and which I hoped to publish, did not gain the encouragement of the public, though I believe the manner I there handled the subject introduced was, in some very scientific quarters, favorably received.

Yours faithfully,

BENJAMIN GOMPERTZ.

KENNINGTON TERRACE, VAUXHALL;
July 15, 1850.

ON PORISMS.

1. With regard to Porisms, I observe, there are many ways in which they make their appearance, in mathematical considerations; sometimes they start, as it were, spontaneously; occasionally they may, in some speculation, be successfully sought after; sometimes they may be invented out of materials of a discussion, which do not of themselves involve them. If any mathematical expression be given, involving given quantities, it will seldom or ever be difficult, in many ways, to introduce arbitrary quantities together with quantities unknown; but to be made known, so connected with the given terms of the expression, that all of such introduced terms, enveloped with the originally contained terms, may be of null effect in the real value of the expression; in consequence of the expression admitting of a development which would cause those introduced terms to vanish. Such introduction of arbitrary quantities, together with the quantities unknown, but to be made known, would turn the expression into a form enunciable as a porism; and we should say, that this altered expression would have the terms, in the first instance stated to be unknown quantities, given, such that whatever the arbitrary quantities were, the expression itself would be given. And here, I may remark, I do

not confine, nor shall I confine, unless I so express myself, my meaning of the word "given" to the meaning attached to it in 'Euclid's Data;' but I wish to imply, by that word, the sense implied by the word determinable, somehow or other.

2. By way of illustration, suppose we had the expression $\sqrt[2]{a+\sqrt[2]{b}}+\sqrt[2]{a-\sqrt[2]{b}}$; a and b being supposed given quantities; and we felt inclined to introduce an arbitrary value ρ, together with an unknown value x to be determined; so that the expression so altered shall remain of the same value as before. Let the altered expression stand $\sqrt[2]{a+\rho x+\sqrt[2]{b-\rho}}+\sqrt[2]{a+\rho x-\sqrt[2]{b+\rho}}$; and state, as a porismatic proposition, that x is given, so that the expression is given whatever ρ may be. If we recollect the original expression, we should at once put this and that equal to each other, and find x, if possible, so that ρ vanishes; but if the expression were proposed to us, without any indication of its origination, we might, for the sake of ease (not of necessity), take some particular case which would facilitate the research, which in my Tracts on 'Imaginary Quantities' I have called "a case of ease," and I should say, if the expression is to be given whatever ρ may be, it will be so, and the same when $\rho=o$; but in that case, it will be $\sqrt[2]{a+\sqrt[2]{b}}+\sqrt[2]{a-\sqrt[2]{b}}$ (the original form), and by making the equality, we shall, by a little reduction, get $2ax\rho+\rho=-2x\rho\sqrt{a^2-b}$, where ρ vanishes, and $x=\dfrac{-1}{2a+2\sqrt{a^2-b}}=\dfrac{-a+\sqrt{a^2-b}}{2b}$. As an example in numbers of the invariable value of the expression, for the same values of a and b, whatever ρ may be. Suppose $a=10$, $b=36$, and x will come out $=-36$, and we should state, that $\sqrt[2]{10-\frac{\rho}{36}+\sqrt[2]{36-\rho}}+\sqrt[2]{10-\frac{\rho}{36}-\sqrt[2]{36-\rho}}$, is always the same, whatever ρ may be; but if $\rho=36$, as a case of ease, it will be equal to $\sqrt{9}+\sqrt{9}=6$; $\therefore$ if the statement be true, it will always be 6, which it is. And this whether the members of the expression be real or imaginary. In this example we have introduced one arbitrary quantity.

3. Suppose, as another example, we had only the simple expression, a, which we wished to "porismatize," (Tracts, i, p. 29) and to have two arbitrary quantities; put $a=\sqrt[2]{k-ht+\sqrt[2]{f-t}}+\sqrt[2]{k-ht-\sqrt[2]{f-t}}$, as one mode of solving the indeterminate problem—square both sides, &c., and we have $a^2-2k+2ht=2\sqrt[2]{\overline{k-ht}|^2-f+t}$; $\therefore a^4-4a^2.\overline{k-h}.t+4f-4t=o$; take $h=\frac{1}{a^2}$; $\therefore a^4-4a^2k+4f=o$ and $f=a^2\left(k-\frac{a^2}{4}\right)$; and we have this Porism, a, being a given quantity, h may be found, so that

$$\sqrt[2]{k-ht+\sqrt[2]{a^2.\left(k-\frac{a^2}{4}\right)-t}}+\sqrt[2]{k-ht-\sqrt[2]{a^2.k-\frac{a^2}{4}-t}}$$

shall be given, whatever k and t may be; and the solution will give $h=\frac{1}{a^2}$; and the expression itself $=a$, whatever k and t may be; so that the value of

$$\sqrt[2]{k-\frac{t}{a^2}+\sqrt[2]{a^2.k-\frac{a^2}{4}-t}}+\sqrt[2]{k-\frac{t}{a^2}-\sqrt[2]{a^2.k-\frac{a^2}{4}-t}};$$

whilst a remains the same, will itself always be the same, and $=a$, the expression proposed to be porismatized, whatever k and t may be; and if, for instance, a were $=3$, this expression would always be equal to 3. But here I should observe, to use the term I employed in my Tracts, it is but a partial, and not an absolute equality, for it is not true with all the roots of the expression; and this observation refers to what has gone before, and to what may be hereafter written.

4. I think I have shown, in my Tract on 'Imaginary Quantities,' the importance of the introduction of the arbitrary quantities, in the explanation of that branch of the mathematics; and among other things in the irreducible case of cubic equations: and as a continuation, or more properly, a more ample illustration, suppose we had the following equa-

tions: $x = z + y$, $x = p + q$, $xy = a$, $pq = m$, $z^t + y^t = b$, $p^t + q^t = b'$. And by the binomial theorem we should have

$$x^t = z^t + y^t + t\,.\,a\,(z^{t-2} + y^{t-2}) + t\,.\,\frac{t-1}{2}\,.\,a^2\,.\,(z^{t-4} + y^{t-4}) + t\,.\,\frac{t-1}{2}\,.\,\frac{t-2}{3}\,a^3\,t\,.\,\frac{t-2}{2}\,.\,\frac{t-3}{3}\,a^3\,(z^{t-6} + y^{t-6}),\ \text{\&c.}$$

$$x^{t-2} = \qquad (z^{t-2} + y^{t-2}) + \frac{t-2}{1}\,.\,a\,.\,(z^{t-4} + y^{t-4}) + \frac{t-2}{1}\,.\,\frac{t-3}{2}\,.\,a^2\,(z^{t-6} + y^{t-6}),\ \text{\&c.}$$

$$x^{t-4} = \qquad \frac{t-4}{1}\,(z^{t-4} + y^{t-4}) + \frac{t-4}{1}\,.\,\frac{t-5}{2}\,a\,.\,(z^{t-6} + y^{t-6}).$$

$$x^{t-6} = \qquad \frac{t-6}{1}\,.\,(z^{t-6} + y^{t-6}).$$

And consequently, $x^t + A\,.\,x^{t-2} + Bx^{t-4} + Cx^{t-6} +$, &c., $= z^t + y^t + (ta + A)\,.\,(2^{t-2} + y^{t-2}) + \left(t\,.\,\frac{t-1}{2}\,a^2 + \overline{t-2}\,.\,aA + B\right)(x^{t-4} + y^{t-4})$, &c., and therefore, if $ta + A = 0$, $t\,.\,\frac{t-1}{2}\,a^2 + \overline{t-2}\,aA + B = 0$, $t\,.\,\frac{t-1}{2}\,.\,\frac{t-2}{3}\,a^3 + \frac{t-2}{1}\,.\,\frac{t-3}{2}\,.\,a^2A + \frac{t-4}{1}\,.\,B + C = 0$, &c., in which case $A = -t\,.\,a$, $B = t\,.\,\frac{t-3}{2}\,a^2$, $C = -t\,.\,\frac{t-4}{2}\,.\,\frac{t-5}{3}\,a^3$, &c., and we shall have, as is well known, $x^t - tax^{t-2} + t\,.\,\frac{t-3}{2}\,a^2\,.\,x^{t-4} - t\,.\,\frac{t-4}{2}\,.\,\frac{t-5}{3}\,a^3x^{t-6} +$ &c. $= z^t + y^t = b$, and we shall have similarly $x^t - t\,.\,mx^{t-2} + t\,.\,\frac{t-3}{2}\,.\,m^2\,.\,x^{t-4} - t\,.\,\frac{t-4}{2}\,.\,\frac{t-5}{3}\,m^3x^{t-6} +$ &c. $= b' = p^t + q^t$, and taking the difference, we have $b' = b + t\,.\,\overline{a - m}\,.\,x^{t-2} - t\,.\,\frac{t-3}{2}\,(a^2 - m^2)\,x^{t-4} + t\,.\,\frac{t-4}{2}\,.\,\frac{t-5}{3}\,.\,(a^3 - m^3)$, and putting $m = a - \rho$, and $P_{\rho, x} = t\,.\,x^{t-2} - t\,.\,\frac{t-3}{2}\,.\,\overline{2a - \rho}\,.\,x^{t-4} + t\,.\,\frac{t-4}{2}\,.\,\frac{t-5}{3}\ \times$

$(3a^2 - 3a\rho + 3\rho^2) . x^{t-6}$, &c., we have $b' = b + \rho . P_{\rho,x}$; but from the equations $p^t + q^t = b'$, and $pq = (m =)\ a - \rho$, we have $p =$

$$\sqrt[t]{\frac{b'}{2} + \sqrt[2]{\frac{b'^2}{4} - \overline{a-\rho}|^t}},\ q = \sqrt[t]{\frac{b'}{2} - \sqrt[2]{\frac{b'^2}{4} - \overline{a-\rho}|^t}}, = \frac{a-\rho}{\sqrt[t]{\frac{b'}{2} - \sqrt[2]{\frac{b'^2}{4} - \overline{a-\rho}|^t}}},$$

whence we obtain, because $x = p + q$, that if $x^t - t . ax^{t-2} + t . \frac{t-3}{2} a^2x^{t-4} - t . \frac{t-4}{2} . \frac{t-5}{3} a^3x^{t-4}$, &c. $= b$, by restoring the value of b', that $x = \sqrt[t]{\left(\frac{b + \rho . P_{\rho,x}}{2}\right) + \sqrt[2]{\left(\frac{b + \rho . P_{\rho,x}}{2}\right)^2 - \overline{a-\rho}|^t}} +$

$$\frac{a-\rho}{\sqrt[2]{\frac{b+\rho . P_{\rho,x}}{2} + \sqrt[2]{\left(\frac{b+\rho . P_{\rho,x}}{2}\right)^2 - (a-\rho)^2}}};$$ whatever ρ may be;

where $P_{\rho,x}$ stands for $t . x^{t-2} - t . \frac{t-3}{2} . \overline{2a - \rho}\, x^{t-4} + t . \frac{t-4}{2} . \frac{t-5}{3} . \overline{3a^2 - 3a\rho + \rho^2}\ (x^{t-6})$, &c.; and as this is so, whatever ρ may be, it is so when $\rho = 0$, and therefore x is $= \sqrt[t]{\frac{b}{2} + \sqrt[2]{\frac{b^2}{4} - a^t}} + \frac{a}{\sqrt[t]{\frac{b}{2} + \sqrt[2]{\frac{b^2}{4} - a^t}}}$;

as is well known, but only mentioned here in connexion with porisms and the case of ease.

But the porism which this investigation presents us with is, that a and b being given, x is given; so that $\sqrt[t]{\frac{b+\rho . P_{\rho,x}}{2} + \sqrt[2]{\left(\frac{b+\rho . P_{\rho,x}}{2}\right)^2 - \overline{a-\rho}|^t}}$

$$+ \frac{a-\rho}{\sqrt{\frac{b+\rho . P_{\rho,x}}{2} + \sqrt[2]{\left(\frac{b+\rho . P_{\rho,x}}{2}\right)^2 - (a-\rho)^t}}}$$ is given, whatever ρ

may be. If, by way of example, we take $t=5, a=1, b=123$, we shall have, for finding of x, the equation, $x^5 - 5x^3 + 5x = 123$, the originating equation of the porism, of which 3 is the only possible value for x — and we should have, in that case, $P_{\rho,x} = 135 - 5 . \frac{5-3}{2} \overline{2-\rho} . 3 = 105 + 15\rho$,

and we should say that

$$\sqrt[5]{\frac{123+\rho.\overline{105+15\rho}}{2}+\sqrt{\left(\frac{123+\rho.\overline{105+15\rho}}{2}\right)^2-\overline{1-\rho}^5}}+\frac{1-\rho}{\sqrt[5]{\frac{123+\rho.\overline{105+15\rho}}{2}+\sqrt{\left(\frac{123+\rho.\overline{105+15\rho}}{2}\right)^2-(1-\rho)^5}}}$$

is given, whatever ρ may be; and as far as the possible root is concerned, will always be $=3$, whether the separate members of the expression are real or imaginary.

5. Had it been proposed to show that x is given, so that the expression

$$\sqrt[5]{\frac{2+\rho.(5x^3-5x(6-\rho))}{2}+\sqrt{\left(\frac{2+\rho.(5x^3-5x.(6-\rho))}{2}\right)^2-\overline{3-\rho}^5}}+\frac{3-\rho}{\sqrt[5]{\frac{2+\rho.(5x^3-5x.6-\rho^2)}{2}+\sqrt{\left(\frac{2+\rho(.5x^3-5x(6-\rho))}{2}\right)^2-\overline{3-\rho}^5}}}$$

shall be given, we should have found that x is given by the equation $x^5-15x^3+45x=2$, having five possible roots, namely, $x=3{\cdot}3080$, $\cdot0443$, $-3{\cdot}2806$, $-2{\cdot}0713$ nearly, and 2 exactly; each of which, substituted in the expression, will make it of the same value, that is to say, respectively $3{\cdot}3080$, $\cdot0443$, $-3{\cdot}2806$, $-2{\cdot}0718$ nearly, and 2 exactly, whatever ρ may be this explains the irreducible case.

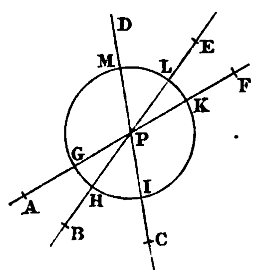
6. Porismatizing is a very common thing in different parts of mathematics, though the operation does not bear that name. For instance, suppose it were required to divide a square number a^2 into two squares: put $a^2=(a-x^2)+\rho^2x^2$, and we shall get $\rho^2x^2-2ax+x^2=0$; and $\therefore x=\frac{2a}{\rho^2+1}$, and we might state as a porism. If a be given, and ρ and x variable, but so related to each other, that when $\rho=0$, x would become $=0$, then x will be a given function of ρ to be found, so that $(a-x)^2+\rho^2x^2$ is given, whatever ρ may be. Then to demonstrate this, take a case of ease, of

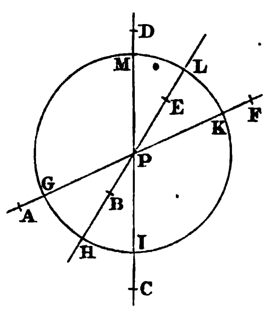
$\rho = 0$, $\therefore$ the expression becomes $= a$, and $\therefore$ if the porism is true, it is always $= a$, and, putting it so, we get $x = \frac{2a}{\rho^2 + 1}$, which answers.

7. A mathematical expression may be a root of an equation to some problem, and by porismatizing such expression, I think it may be said, that we are porismatizing the problem; but this may also be done *without eliciting the root*. Suppose, for instance, we had, resulting from some problem, $\phi(a, b, c, \&c.)\, x^2 - \phi'(a, b, c, \&c.)\, x + \phi''(a, b, c, \&c.) = 0$. Where ϕ, ϕ', ϕ'' are known fractional characteristics of the quantities a, b, c, &c.; a, b, c, &c. being supposed independent elements of the problem; x being the quantity sought after in the independent or general state of the problem. Then, if such relationship among some of the values a, b, c, &c. can be assumed, that a group of the others may vanish from all the said functions, characterised by ϕ, ϕ', ϕ'', x will be determinable, independently of that group, and the problem might be said to be porismatized with respect to that group, and relationship assumed among the vanishing elements of that group. And we would say, for instance, that such values, or such state of relationship, could be assumed among certain of the elements a, b, c, &c., or such coefficients be given to a group, that the whole of the group would vanish from the equation, and leave x the same whatever they might be.

8. By way of simple illustration, suppose a, b, c, m, n, were elements of the problem requiring the value of x in the equation $x^2 - (a^2 + ab - \overline{2a - b}.m + 3.\overline{b - c}.n)x + ab = 0$; then it is evident, that if $2a$ were taken $= b$, and $3b = c$, consistently with the problem, the equation would become $x^2 - (a^2 + ab)x + ab = 0$; or, if we please, $x^3 - 3a^2x + 2a = 0$; and then for x would be given whatever m and n were, and stating this case as a porismatic proposition, we might state, that if one element a, of the problem, were given, the two other elements b, c may be found, so that the value sought by the problem would be given, whatever the other elements m, n of the problem were. The following example, I think, may be added with advantage, to illustrate the observations of this article.

9. Suppose $\overline{a-x}\,.\,\overline{b-x}+\overline{c-x}\,.\,\overline{d-x}=\overline{e-x}\,.\,\overline{f-x}+c\,.\,d+h$, and therefore $x^2-(a+b+c+d-e-f)\,x+ab-ef-h=0$; then if $a+b=e+f$, and $ab=ef+h$, the equation will become $x^2-\overline{c+d}.x=0$; whatever a, b, e, f may be, provided $a+b=e+f$, and $ab=ef+h$, as stated. Now, the general equation may be an equation resulting from the following problem: Three right lines, AF, BE, CD, intersecting each other in a common point P, having the segments $AP=a$, $PF=b$, $PB=c$, $PE=d$, $CP=e$, $PD=f$, about P as a centre, to describe a circle $GHIKLM$, cutting AF in G and K, BE in H and L, CD in I and M, so that $AG\,.\,KF+BH\,.\,LE=CI\,.\,MD+PB\,.\,PE+$ a given magnitude h. And this problem, if x be put for the radius of the circle, evidently gives immediately the above equation; and the porism it offers may be that, BE being given, that two equations expressing the relationship between the five elements, AP, PF, CP, DP, and h, may be found, so that the radius of the circle about P shall be given, which will cause $AG\,.\,KF+BH\,.\,LE$ to be $=CI\,.\,MD+PB\,.\,PE+$ the magnitude h; and that there will be five arbitrary quantities. Supposing nothing given but the two equations. The angles are not included in these five, which are, of course, arbitrary. All that is necessary is to make, for instance, $AF=DC$, and radius $\underset{\cdot}{=}BE$, supposing h not given, but to be found.





10. Moreover, if one or more of the elements a, b, c, &c. of a problem, by assuming certain conditions among some of them, should vanish from the equation, a circumstance which might arise from the coefficients of the vanishing elements becoming separately equal to nought, in conse-

quence of the conditions assumed among the other elements; or, because they vanish in groups by the assumption; thereby constituting separate equations among the vanishing elements. As, supposing, for instance, the elements m, n, p, q vanish by certain restricted relationships assumed among the remaining elements, which give one equation between m and n, and one equation between p and q, &c., by pairs; the circumstance would, or might give rise to so many (to use the expression) analytical plane loci, or, possibly, real graphical plane loci, accompanied with porismatic properties. If the conditions gave, for instance, one equation between m and n, and one between n and p, it would or might give rise to a graphical, or else (to use the term) an analytical, double curvature locus, connected with porismatic properties. And if the conditions, as far as m, n, p were concerned, only give one equation between m, n, and p, they might give rise to some superficial locus connected with porismatic properties; and under these circumstances, the porism might, I think, be appropriately called Local Porisms, or Porismatic Loci; a term I had adopted between forty or fifty years back; and, in some cases, perhaps, the term Multiple Local Porisms might not be inappropriate.

11. I further observe, with respect to the equation $\phi(a, b, c, \&c.)x^2 - \phi'(a, b, c, \&c.)x + \phi''(a, b, c, \&c.) = o$; or, with respect to higher or lower equations, that, though we have been treating x as the unknown quantity of the problem, in its general form; and that x is the quantity which is proposed to be shown to be given under changes of some of the elements, in consequence of certain conditions being to be found and assumed among other of the elements of the problem, I observe, that if we could assume such conditions among the elements, that we should at once have $\phi(a, b, c, \&c.) = o$, $\phi'(a, b, c, \&c.) = o$, and $\phi''(a, b, c, \&c.) = o$, provided those functions do not become all nought in consequence of the introduced conditions first causing them to have a common multiplier, which, by additional conditions, becomes equal to o, x would be indeterminate. And as a porism might be derived from it, we see, therefore, why a porism is said to be derived from the indeterminate case of a problem. But this

porism is derived evidently, as far as x is concerned, from the proposed equations in a way different from the other, though the equation, by considering a different element to be the sought element, might be put in that form as to bring the case into similar circumstances as the previous case is in.

12. But porisms are derivable from the elements of a problem in another way. Suppose x were the element of a problem inquired after, in virtue of an equation between it and the other elements $a, b, c, m, n, p,$ &c., and that the equation for determining x were $x^t - \phi(a, b, c, \&c.)x^{t-1} + \phi'(a, b, c, \&c.)x^{t-2} - \&c. = 0$; and suppose that all the roots of this equation to be possible, then if the coefficient of x^{t-1}, namely $\phi(a, b, c, \&c.)$ did not contain m, n, p; and the other elements in the coefficient $\phi(a, b, c, \&c.)$ were all given, then as the sums of all the roots of the equation would, from the nature of equations, be $= \phi(a, b, c, \&c.)$, and which, not containing $m, n, p,$ would remain the same, whatever $m, n, p,$ the other elements of the problem might be. And this equation between the sum of the roots and the elements, not including $m, n, p,$ would be a general theorem allied to a porism; but if, under the consideration of $a, b, c,$ &c. being given, the said sum were stated to be given, or, more properly, stated to be of a discoverable value, then the proposition would be still nearer allied to a porism; but it would be deficient with respect to a porism, as though it would state, that notwithstanding the variability of some of the elements of the problem, the said sum of the roots would be given, or might be required to be found, it would not seek or require any alteration to be made among the elements, to constitute that property; and probably such a proposition might not improperly be called a deficient porism. But if the function $\phi(a, b, c, \&c.)$, in its general state, contain $m, n, p,$ but by some introduced conditions required to be shown, should become disencumbered with $m, n, p,$ then such a proposition fully announced, requiring such conditions to be shown, would, according to my comprehension of the word, be a porism. These remarks will apply to the other coefficients of the equation to the problem, and to expressions

generally; and if the roots in the above case were x', x'', x''', we might state as a porism, that such relationship could be found in the terms of the function $\phi(a, b, c, \&c.\ m, n, p)$ as would cause $x'+x''+x'''$ &c., to be given, independently of m, n, p considered as arbitrary. A similar remark refers to $x'x''+x'x''$ &c. $+x''x''$, &c.

13. And, moreover, if any expression can be found between the coefficients of x^{t-1}, x^{t-2}, &c., which shall exclude m, n, p, that expression if it leave all the other elements a, b, c, &c. independent of each other, will, made-up of sums, sums of rectangles, &c., give a general theorem allied to a porism; and if the remaining elements are given quantities, and the inquiry be to find how to compose the function of the coefficients of x^{t-1}, x^{t-2}, &c., so that such a function, notwithstanding the variability of certain of the elements of the problem, shall become given, and be required to be shown, we shall have a porism.

14. And besides these modes of discovering porisms, I may add, that if

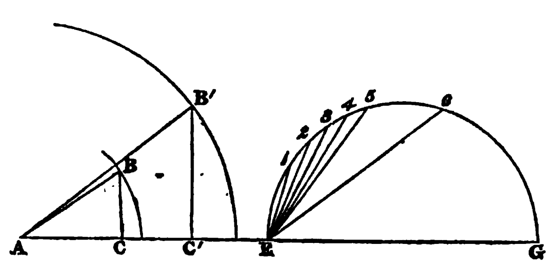
$$x=f(a, b, c, \&c.\ r),$$
$$y=f_{,}(a, b, c, \&c.\ r),$$
$$z=f_{,,}(a, b, c, \&c.\ r), \&c.;$$

each function containing a common element r, the functional characteristics $ff_{,}f_{,,}$ being of the same or of different character. If we could assume $\phi(x, y, z, \&c.)$, so that when by substituting the values for x, y, z, &c., the resulting function should not contain r, we might obtain a general proposition allied to porisms, a defective porism, or a perfect porism as the case might be; and so with regard to more than one element which may be got rid of.

15. And it is noticeable, that a porismatic proposition may require one or many things to be found, so that one thing may be shown to be given, or that many things may be shown to be given, and that one or many conditions or elements may be arbitrary.

16. Take, for example, the equation $2x^2-gx=a^2+b^2+c^2+d^2-e^2-f^2$, which would result from the following geometrical problem: x is the

diameter of a circle to be found, a, b, c, d, e, f are chords in the circle given in magnitude of arcs, which I will call A, B, C, D, E, F respectively, to find x, so that the sum of the versed sines of the supplements of the arcs $ABCD$, shall be equal to the sum of the versed sines of the arcs, which are the supplements to E and F, together with a given magnitude g. This problem evidently gives immediately the equation $\frac{x^2-a^2}{x}+\frac{x^2-b^2}{x}+\frac{x^2-c^2}{x}+\frac{x^2-d^2}{x}=\frac{x^2-e^2}{x}+\frac{x^2-f^2}{x}+g$; and consequently the equation $2x^2-gx=a^2+b^2+c^2+d^2-e^2-f^2$. Now I observe, that if e and f be given, and a, b, c, d, instead of being given, are so far arbitrary, that the value of each depends on the two equations $a^2+b^2=e^2$, $c^2+d^2=f^2$, but are not beyond them restricted, and we shall have $2x^2-gx=0$, and consequently x given; whatever a, b, c, d are, provided they be subject to the said two equations. Hence we may get this double local porism. A point A and a right line AG, even given in position, and three magnitudes g, e, f being given, then if about A as a centre, two circles be described, the one with radius $AB=e$, and the other with the radius $AB'=f$, are from any points B, B' in the respective circles, $BC, B'C'$ be drawn $\perp AG$ cutting-in in C and C' respectively, then the circle will be given in magnitude, such that if chords be taken in it of the magnitudes AC, BC, AC', $B'C$, AB, AB', the sum of the versed sines of the supplement of the arcs, of which the first four are the chords, shall be equal to the sum of the versed sines of the supplement of the arcs whose chords are AB, AB', together with a given magnitude g. And not only may two out of the four quantities AC, BC, AC', BC' be arbitrary, but even if the radii AB, AB' be arbitrary, the said diameter will be given.

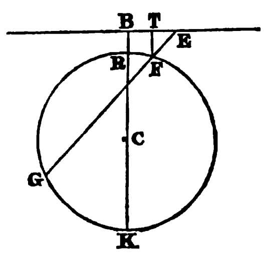


17. Moreover, referring to the equation $2x^2 - gx = a^2 + b^2 + c^2 + d^2 - e^2 - f^2$, suppose some other magnitude k were given, and that instead of a, b, c, d, e, f, being given, we had only the equations $e^2 + f^2 = d^2$, and $a^2 + b^2 + c^2 = k^2$, then the equations would become $2x^2 + gx = k^2$, and, therefore, as g and k are given, x will be given, whilst the two quantities e and f are only subject to be the corresponding sine and cosine of any arc whose radius is d, which is itself arbitrary. And a, b, c are the rectangular co-ordinates of any point, in a sphere whose radius is given $= k$, and whose centre is the origin of the co-ordinates: and this gives what perhaps may be called a compound local porism, resulting from a plane locus and a superficial locus. Reverting again to the same equation $2x^2 - gx = a^2 + b^2 + c^2 + d^2 - e^2 - f^2$, suppose we had $c = a$, $a^2 + b^2 = e^2$, $c^2 + d^2$, that is $a^2 + d^2 = f^2$, then would a, with its equal c, b, d, be definable by the rectangular co-ordinates of a line of double curvature, formed by the intersection of two right cylinders, whose axes are perpendicular to each other, on circular bases, whose radii are themselves arbitrary, and the intersection of whose axes are the origin of the co-ordinates.

And here, with regard to definitions and terms to be used, I remark, that when a geometrical problem is proposed to be solved by means of symbols, it may not be improper to denominate the operation with which the solution commences, symbolizing the problem. But the reverse of this when we wish to give geometrical examples of the use of operations which had only been symbolically considered, may often be an interesting inquiry, and lead to a good purpose, though the problem to do this is completely an indeterminate problem, as it admits of infinite diversity. This problem may, I think, be aptly called a Problem to geometricaize symbols. The application of this is evident in this article, and in what follows, and I will remark, that the introduction of new terms, as these and others are, which I have ventured to introduce, I cannot consider of little importance, for they may involve hints of important theories and researches, and so, I think, among such introductions, I may say the words, to porismatize,

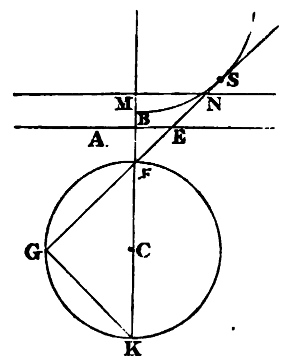
partial equality, conditional equality, and absolute equality, introduced in my Tracts on 'Imaginary Quantities,' contain hints of wide speculation worthy of attention.

18. Porismatize geometrically $g \, . \, h$. One way of doing this may be



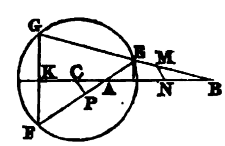
the following: suppose G the centre of a circle GRF, whose radius is $\frac{1}{2}h$, and that the right line $KCRB$ cuts the circle in R and K: take B in the said right line, so that $BR = g$, and $BR \, . \, RK$ will be $= g \, . \, h$; draw a right line $ABTE$ through $B \perp BRCK$, take in it a fixed point A, let F be some point to be found in the circumference of the circle, so that some right line GFE, cutting the circle in G and F, and the right line AB in E shall be so cut that $GF \, . \, FE$ shall $= g \, . \, h$. Draw $FT \perp ABE$ cutting it in T, take $AB = b$, $AT = x$, $TF = y$, $AE = \rho$; and by an easy process we shall find $EF \, . \, FG$, or its equal by hypothesis gh, $= -2\rho b + b^2 + g^2 + hg + 2\rho x - x^2 - y^2$, (where $(x-b)^2 + \overline{g + \frac{1}{2}h - y}^2 = \frac{1}{4}h^2$). This equation gives $x^2 + y^2 - \overline{2 \, . \, x - b} \, . \, \rho - (b^2 + g^2) = 0$; and we at once see, that if $b = x$, that is, if F coincide with R, that ρ vanishes, and that, considering b given, x and y will be given; and we have Simson's porism, namely, "having a right line, AB, given in position, and a circle given in position and magnitude, the point will be given through which if a right line be any ways drawn, meeting AB and the circle, we shall have the rectangle between the segments of the line drawn between the points sought and the points of concourse given."

This porism offers an immediate solution to the following problem: A circle being given in position (and magnitude), and a right line MN in position, to draw, if possible, two right lines AE, EG, intersecting each other in E, the one AE being parallel to a line MN given in position, and the other GE touching a given locus, and cutting a given circle in F and G, so that the rectangle of EF and EG shall be equal to a given magnitude.



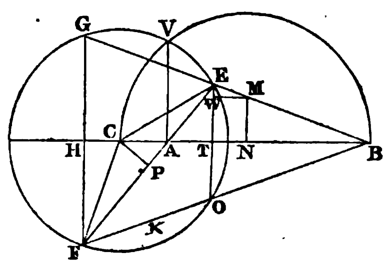
19. And the construction which this porism immediately points out, is through the centre C, of the given circle, draw $CL \perp MN$, cutting the circle in F and K, in CL; take B, so that $BF \cdot FK$ shall be equal to the given rectangle; draw AB through B, a parallel to MN, and through F draw the right line GFE to touch the given locus (granting we have the means of so doing, either as a plane or a solid problem). If the locus were a circle, the method of doing it, I observe, is known; and the thing is done, the reason of which I think will be evident. But for our satisfaction, I will, independently of the above porism, analyse the problem, so as to give the same composition.

Suppose it done, and that GFS cuts the circle in G and F, and touches the given locus, and that AE is parallel to the given line MN, through the centre, C, of the circle, draw $CFB \perp AB$, and $\therefore \perp MN$ cutting AB in B, and the circle in F and K, and join GK, and $\therefore$ because B and G are right-angles $BF : EF :: FK : GF$; $\therefore BF, FK =$ the given magnitude $EF \cdot FG$, and consequently because KFB the right line and the circle are given in position, and $\therefore K$ and F given, B is also given; $\therefore AB$ is given in position, it being $\parallel MN$ given in position; and as F is given, and GFE touches a given locus GFE, it is given in position, Q. E. I. And this affords the same composition as that given above.



20. This is also a problem leading to a porism, C is the centre of a circle $FOEG$ given in position and magnitude, A a point given within it, EAF a right line, cutting the circle in E and F; FG a right line $\perp$ to the right line passing through C and A, and cutting the circle again in G; required the equation of the line GE? Let M be a point in the line whose equation is required, $MN \perp$ the line passing through C and A cutting it in N; draw $ET \perp CATN$, and $WM \perp EWT$; put $CE = r$, $CA = a$, $CP \perp$ right line $EAPF = \rho$, $AN = x$, $MN = y$.

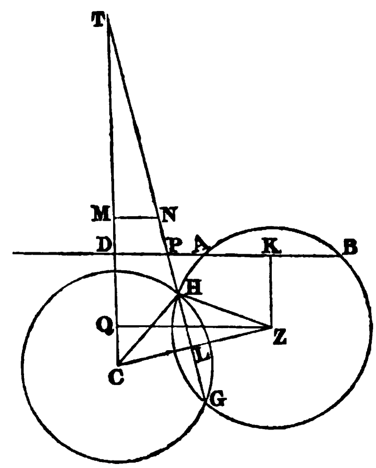
Then, by an easy process, we obtain for the equation of the line *GEM*, $\rho \cdot \frac{r^2 - a^2}{a} - y\sqrt[2]{\rho^2 - x^2} = \rho x$, and as a corollary we see that if $y = o$, that is if M be in the concourse B, or if the two right lines GE, CA, we shall have $x = \frac{r^2 - a^2}{a}$, whatever ρ may be. Draw $AV \perp CA$ cutting the circle in V, and it is evident that CVB is a right-angle, in consequence of this equation; and that V will be in the circle whose diameter is CB, and that if O be the intersection of BF, and the given circle, FO, GE, cut off by the lines passing through BE, BF are equal; and therefore we immediately get Simson's porism, prop. 50, p. 463,—"That a circle being given in position and magnitude, and a point A in position, and a right line EAF be anyhow drawn, cutting the circle in E and F, then a point will be given through which two right lines being drawn through the points EF of intersection of the circle with EAF, or GAO, they will cut off equal arcs." And this whether the point be in or out of the circle, because if B be the point given, A will be found, from an evident property in the above solution; but if B be given, the proposition would stand thus; that if through B any right line BEG be drawn, cutting the circle in E and G, then a point A will be given, through which two right lines GO, EF being drawn, they will cut off equal arcs. And the composition for finding A, is to draw a semicircle on the diameter of CB, cutting the given circle in V, and drawing $VA \perp CB$ cutting it in A.



21. As it may be interesting frequently to see the use of porisms, I remark that this porism will afford a ready solution to the problem. "In a given circle to inscribe a trapezium whose diagonals, if possible, shall intersect in a given point, two of its opposites being equal, and one of the same opposite sides passing through a given point." The solution of this is

evident, taking A the point in the above figure through which the diagonals pass. KA, the point through which one of the equal opposites are to pass; because by the porism B is given, and therefore BF, &c., are given.

Another problem tending to a noted porism published in the 'Mathematical Repository,' by my early friend, the erudite and ingenious Mr. Mark Noble.



22. Two circles being given in position and magnitude, to find the equations of a right line passing through the two points of their intersection.

Let C be the centre of the circle whose radius is r, Z that of the circle whose radius is ρ, CDI any line at pleasure passing through C, cutting the line GH which passes through the two intersections G and H of the two circles in I; let MN be $\perp$ CI, cutting CI in M, and GI in N be $= y$, and let $CM = x$; draw DAB at pleasure $\perp$ CI cutting CD in D, the circle whose radius is ρ, in A and B, and the right line GHI in P; parallel to DB draw ZQ cutting CI in G; draw $ZK \perp DB$ cutting it in K; and let the right line CZ cut GHI in L. Put $CD = a$, $DB = b$, $AB = c$; $\therefore ZK = \sqrt[2]{\rho^2 - \frac{c^2}{4}}$, $CZ = \sqrt[2]{(a - \left(\rho - \frac{c^2}{4}\right)^{\frac{1}{2}})^2 + \overline{b - \frac{1}{2}c}^2} = \sqrt[2]{a^2 + b^2 - bc + \rho^2 - 2a\sqrt[2]{\rho^2 - \frac{c^2}{4}}}$; $CH^2 - CL^2 = HZ^2 - LZ^2$; $\therefore r^2 = \rho^2 - LZ^2 + CL^2 = \rho^2 - CZ^2 + 2LC \,.\, CZ$; $\therefore LC = \frac{r^2 - \rho^2 + CZ^2}{2 \,.\, CZ}$: and as $CQ : CZ :: CL : CI$; $\therefore CI = \frac{CI \,.\, CZ}{CQ} =$

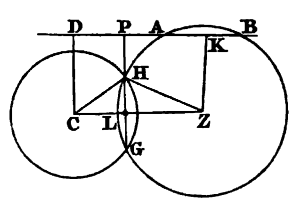
$$\frac{r^2 + a^2 + b^2 - bc - 2a\sqrt[2]{\rho^2 - \frac{c^2}{4}}}{2 \,.\, \left(a - \sqrt[2]{\rho^2 - \frac{c^2}{4}}\right)}.$$

But $MI = CI - x : MN = y :: ZO : CQ$;

$$\therefore CQ.CI - CQ.x = y\,(b - \tfrac{1}{2}c), \text{ that is } \frac{r^2 + a^2 + b^2 - bc - 2a\sqrt{\rho^2 - \frac{c^2}{4}}}{2.(b - \frac{1}{2}c)} - \frac{a - \sqrt{\rho^2 - \frac{c^2}{4}}}{b - \frac{1}{2}c}.\,x = y,$$ the equation of the line GH, which was sought.

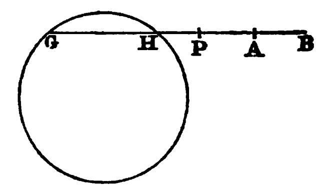
23. Corollary, if $x = a$, the factor $\sqrt{\rho^2 - \frac{c^2}{4}}$ vanishes, and we have $y = \frac{r^2 - a^2 + b^2 - bc}{2.(b - \frac{1}{2}c)}$; in which case N will coincide with P, and $y = DP$; and therefore if r, a, b, c, be given, DP will be given; and therefore the point P, whatever ρ, the radius of the circle HAB, may be. And this gives the porism above alluded to published by my early friend, Mr. Noble; namely, a circle being given in position and magnitude, and two points being given, a point P will be given in the line which passes through the two points of intersection of the given circle, and any circle described through the two points A and B.

24. And to illustrate a remark I have made with respect to the demonstration of porismatic propositions, relative to what I have termed cases of ease, to be assumed, and to give examples of the sufficiency of assuming extreme and impossible cases, I give the following process, though a much easier way may be used of obtaining a demonstration.

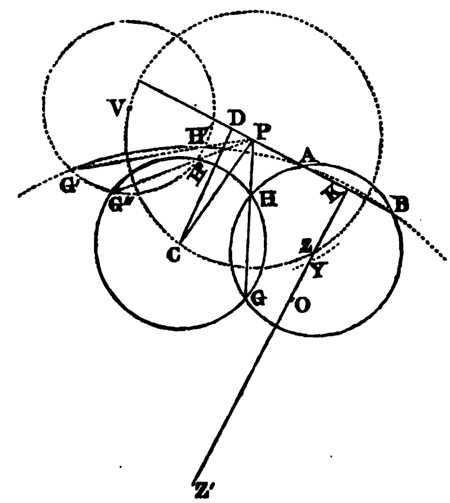


Suppose the porism true, and take Z in the line passing through C ∥ the line AB; then the line passing through G and H will be $\perp$ CZ; and $HZ = AZ = \sqrt{a^2 + \frac{1}{4}c^2}$ is given; and L being the intersection of HG and CZ, CL is given, and is, by the bye $= \frac{b^2 - bc + r^2 - a^2}{2(b - \frac{1}{2}c)}$. And further, for another case of ease, suppose the radius of the circle chosen, which passes through A and B, to be infinite, or, in other words, that the circle becomes a right line, then the intersection

of this infinite circle with the given circle will be in the right line BA itself; and though this case would not be generally possible, though it might be so; for BA in the above figure does not cut the given circle, still it must lead to the finding of P, the point said to be given, if the porism be true; but in the last case of ease it was shown to be in a line passing through $L \perp DP$, given in position; consequently, if the porism be true, P must be in the intersection of LP and AB of the former figure, and DP will be $=\dfrac{b^2-bc+r^2-a^2}{2(b-\frac{1}{2}c)}$, and this point, by proceeding with the analysis, will be found to answer for any radius ρ, and therefore the porism is true.



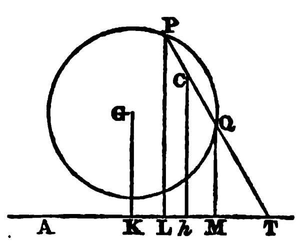
25. I now observe, that as $DP=\dfrac{r^2-a^2+b^2-bc}{2.(b-\frac{1}{2}c)}$, that $BP=\dfrac{a^2+b^2-r^2}{2(b-\frac{1}{2}c)}$, and putting this $=g$, we shall have $a^2+b^2-r^2=2bg-cg$; or $a^2+(b-g)^2$ or its equal $PC^2=r^2-cg+g^2$, and therefore if PC, c and g are given, r is given, and $r^2=PC^2+\frac{1}{4}c^2-\overline{g-\frac{1}{2}c}|^2$, and we may get this *Local Porism*, namely, that three points, B, A, P, in a right line being given, and a magnitude PC be given, if a circle be described about P, with a radius PC, it will be the locus of the centres of all circles whose radius r will be given, such that if any circle be described through A and B, cutting any of the circles described with the radius r, whose centre is in the said locus of centres, the line passing through the two points of intersection will pass through the one and the same point which will be given. And the composition for the finding of the radius r, of these innumerable



circles, may be to take O in the line $\perp$ AB, which bisects it in K, so that $KO=PC$, and with AO as radius, and centre P, strike an arc cutting KO in Y, and KY is $=$ the radius r, of the innumerable circles whose centre is in the locus of the circle described about P, with the radius $=PC$, each of which have the porismatic property of cutting every circle passing through A, B, so that the right line passing through the two points of intersection of any of these circles with any of the others, always passes through a point which will be given in position, and which point is P. And because the line passing through the two points of intersection of any two equal circles, will pass through the centres of any circle which passes through the centres of the said equal circles, the line $G''H''$ of any of the two equal circles will also pass through P. And I hint, that one use of this porismatic locus may be to solve the following problem.

26. Given three points, B, A, P, in a right line, and a circle about P being also given. To describe a circle whose centre is in the aforesaid circle, which shall touch a line given in position, and which shall intersect every circle passing, when such intersections are possible, through A and B, in two points, that a right line passing through the two intersections of the circle to be found, and any one of the other circles, shall pass through the point P.

27. The following is a problem furnishing porisms of interest, I insert it as an example: G is the centre of a circle given in position and magnitude, AT, CT are two right lines given in position, Q is the intersection of the circle with the line TC, and QM is $\perp$ AT cutting it in M. QM is required. Let A and C be given in points respectively in the lines AT, CT; G the centre of the circle, $GK=q$, $ch=b$, $QM=y$ perpendiculars to AT, cutting it respectively in K, h, and M, $\angle ATC=\rho$, $AK=p$, $Ah=a$, radius of the circle r, then $KM=\sqrt{r^2-\overline{q-y}|^2}$, $Ch-QM=b-y$, $hM=\overline{b-y}.\text{cotangent of }\rho$; but hM is also $=AK+KM-Ah=p+\sqrt{r^2-\overline{q-y}|^2}-a$,



consequently putting these two values of hM equal to each other, we evidently get $(b-y)$ cot of $\rho+(a-p)=\sqrt{r^2-(q-y)^2}$, and, consequently, by reduction, putting $A=2.\left(\frac{q-\overline{a-p}\,.\,\text{cot of }\rho+b\ \overline{\text{cot of }\rho)^2}}{\text{cot of }\rho^2+1}\right)$, and $B=\frac{b^2.\overline{\text{cot of }\rho)}^2+2.\overline{a-p}.b.\text{cot of }\rho+\overline{a-p)}^2-r^2+q^2}{\text{cot of }\rho^2+1}$, we have $y^2-Ay+B=0$. Now I observe, that if CT cut the circle again in P, and we draw $PL\perp AT$ cutting it in L, by the nature of equations A is $=QM+PL$, and $B=QM\times PL$, and as A does not contain r (the radius of the circle), that we have this general theorem, that if the centre G of any circle be given in position, and two right lines AT, CT be given in position, and CT cut the circle in P and Q; and PL, QM perpendiculars to AT cut AT respectively in L and M, whatever the radius r may be we shall always have $PL+QM$ given, this theorem, if stated as a proposition, would contain a porismatic quality, but it would not be a porism. But if there be another right line $P'Q'T'$ parallel to PT, cutting AT in T' and the circle in P' and Q', passing through a given point C'; and $P'L'$, $C'k'$, $Q'M'$ be drawn $\perp AT$ cutting AT respectively in L', k', M' (which lines and points, to save confusion in the diagram, I have not placed in it), and if $C'k'=b'$, $Ah=a'$, $A'=2.\left(\frac{q-(a'-p).\text{cot of }\rho+b'.\overline{\text{cot of }\rho)}^2}{\text{cot of }\rho^2+1}\right)$, $B'=\frac{b'^2.\text{cot of }\rho^2+2.\overline{a'-p}.b'.\text{cot of }\rho+\overline{a'-p)}^2-r^2+q^2}{\text{cot of }\rho^2+1}$; and if K be any numerical value, we shall have $QM+PL-K.(Q'M'+P'L')=A-KA'=\frac{2}{\text{cot of }\rho^2+1}\times(q.\overline{1-k}+\overline{p-a-pk+a'k}.\text{cot of }\rho+\overline{b-kb'}.\text{cot of }\rho^2)$, $\therefore$ whatever ρ the value of the $\angle\angle$ which CT, $C'T'$ make with AT; provided $q.\overline{1-k}=b-kb'$, and $p-a-pk+a'k=0$, we shall have $QM+PL-k.(Q'M'+P'L')=2q.\overline{1-k}$; that is, if $q=\frac{b-kb'}{1-k}$, and $p=\frac{a-ka'}{1-k}$; making $QM+PL-k.(Q'M'+P'L')=$

$2b - 2k.b'$, whatever ρ may be, and we have this Porism. Two lines being drawn parallel to each other through two given points (that is to say, one through one given point, and the other through the other given point), cutting a right line given in position, the centre of a circle is given, whose radius is arbitrary, and whatever the angle may be at which the two parallel lines passing through their respective given points, cut the right line given in position, such that the sum of the perpendiculars demitted on the line given in position, from the two points, where one of the parallel lines cuts the circle comprehended between those points and right line given in position $-k$ times, the perpendiculars demitted on the said given line from the two points where the other line cuts the circle comprehended between those points and said line given in position, is a quantity which will be given. And I note, that when $k = 2$, this will be the porism in Dr. Brewster's 'Encyclopædia,' under the word "Porism" (page 112), in the excellent paper which, the Doctor informs us, was written by the celebrated Mr. Babbage, and on which account this problem is brought as an example of my views.

28. And to continue from the same problem, and with regard to the same excellent paper, to illustrate my view: Looking to the last term of the equation $y^2 - Ay + B = 0$ in which

$$B = \frac{b^2.\text{cot of } \rho^2 + 2.\overline{a-p}.b.\text{cot of } \rho + (a-p)^2 - r^2 + q^2}{\text{cot of } \rho^2 + 1} \text{ represents}$$

$PL \times QM$. I observe, that if a line be drawn through $C \perp$ to the former line PQ, and that line cuts the circle in $P_,$ and $Q_,$, and we write cotan of $\overline{\rho + 90^\circ}$, that is $-\dfrac{1}{\text{cotan of } \rho}$ in the expression of B for cotan of ρ we shall get $P_,L_, \times Q_,M_, =$

$$\left(\frac{b^2}{\text{cot of } \rho^2} - 2.\overline{a-p}.\frac{b}{\text{cot of } \rho} + (a-p)^2 - r^2 + q^2\right) \div \left(\frac{1}{\text{cotan of } \rho^2} + 1\right),$$

or its equal $\dfrac{b^2 - 2.(a-p)\,b.\text{cot of } \rho + \overline{(a-p)^2 + q^2 - r^2}.\text{cot of } \rho}{\text{cot of } \rho^2 + 1}$, and

therefore, if we wish to have $PL.QM = P_{,}L_{,}.Q_{,}M_{,}$ we must make this rectangle equal to the former, which will, by equating them, show that it requires the conditions that (cot of $\rho^2 - 1) \times \left(r^2 - q^2 + (a-p)^2 - b^2 - 4.\overline{a-p}.\text{cot of } \rho\right) = 0$; and $\therefore$ we must have, for ρ to be indeterminate $a = p$, and $b^2 = r^2 - q^2$, which will satisfy the condition, and therefore we have this Porism, given on page 110, in the same work of Mr. Babbage: "A circle and a straight line being given in position, a point may be found within the circle, such that if any two lines be drawn through that point, at right-angles to each other, the rectangle under the perpendiculars let fall from the extremities of one chord to the line given in position, shall be equal to the rectangle of the perpendiculars let fall from the extremities of the other chord on the same line."

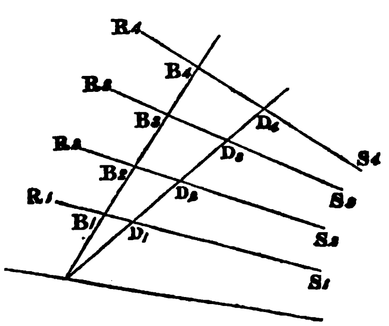
29. But to show how one and the same problem may lead to general propositions allied to porisms, and to porisms, and to porismatic loci. Suppose some other point $\overline{C}$ given, and that two lines $\perp$ to each other be drawn through it $\perp$ each other, and that the one makes the $\angle, \bar{\rho}$ with the given line, and cuts the aforesaid circle in $\overline{P}$ and $\overline{Q}$, and the other cuts that circle in $\overline{P}_{,}$ and $\overline{Q}_{,}$, and that lines be demitted from these points $\perp$ to the given line cutting it in $\overline{L}$, $\overline{M}$. $\overline{L}_{,}$, $\overline{M}_{,}$ respectively, then looking to the values of the rectangles $PL.QM$, $P_{,}L_{,}.Q_{,}M_{,}$ above given, we observe that

$$\left.\begin{array}{l} PL.QM + P_{,}L_{,}.Q_{,}M_{,} = q^2 + (a-p)^2 + b^2 - r^2 \\ \overline{PL}.\overline{QM} + \overline{P_{,}L_{,}}.\overline{Q_{,}M_{,}} = q^2 + (\bar{a}-p)^2 + \bar{b}^2 - r^2 \end{array}\right\}$$

$\bar{a}$ and $\bar{b}$ standing in lieu of a and b for the co-ordinates of $\overline{C}$; $\rho, \bar{\rho}$ not being contained in this expression. With regard to the first of these, I observe (and it would equally, *mutatis mutandis*, refer to the second), that it affords a general theorem, so far allied to a porism as, that $PL.QM + P_{,}L_{,}.Q_{,}M_{,}$ remains of the same value, however the $\angle\,\rho$ may vary. But if $q^2 + (a-p)^2 + b^2 - r^2$ should be $= k^2$, and a, b, r, k be stated to be given, then the locus of G the centre of the circle, will be restricted to a circle whose radius $k^2 - b^2 + r^2$ is given. And we are furnished with the following porismatic locus. That if a point C and a right line be given, and

a magnitude r be given; and if about a centre taken anywhere in the circumference of a locus, with the radius $= r$ circles be described, and any two right lines perpendicular to each other, passing through a given point C, the one cutting any of these circles in P and Q, and the other cutting the same circle in $P_{,}$ and $Q_{,}$, and PL, QM, $P_{,}L_{,}$, $Q_{,}M_{,}$ be drawn $\perp$ the given line cutting it respectively in L, M, $L_{,}$, $M_{,}$, and make $PL.QM + P_{,}L_{,}.Q_{,}M_{,} =$ a given quantity k, then I say the locus will be given such that if any other two lines $\perp$ each other, passing through the given point C, cut any of these circles, the one in P'', Q'', and the other in $P_{,}''$, $Q_{,}''$, and the lines $P''L''$, $Q''M''$, $P_{,}''L_{,}''$, $Q_{,}''M_{,}''$ be drawn $\perp$ to the given line, cutting it respectively in $L''M''$, $L_{,}''M_{,}''$, that $P''L''.Q''M'' + P_{,}''L_{,}''.Q_{,}''M_{,}''$ shall be $=$ the same given quantity k. Moreover, by taking the difference in the two equations in the brackets, we have $PL.QM + P_{,}L_{,}.Q_{,}M_{,} - \overline{PL}.\overline{QM} - \overline{P_{,}L_{,}}.\overline{Q_{,}M_{,}} = a^2 - \bar{a}^2 - 2.\overline{a-\bar{a}}.p + b^2 - \bar{b}^2 = b^2 - \bar{b}^2$. If $a = \bar{a}$, whatever r, a, p, q, ρ_2 ρ may be, that is to say provided the two points C, $\underline{C}$ be in a right line $\perp$ to the given right line, at the distance b, $\bar{b}$, therefore the said expressions will be given.

30. The following problem leading to a porism, may be a good example of my views. A point A, and four right lines, R_1S_1, R_2S_2, R_3S_3, R_4S_4, being given in position in a plane, to draw a right line $AB_1B_2B_3B_4$ through A, cutting those four right lines respectively in B_1, B_2, B_3, B_4, so that $AB_1 : AB_2$ as $AB_3 : AB_4$.

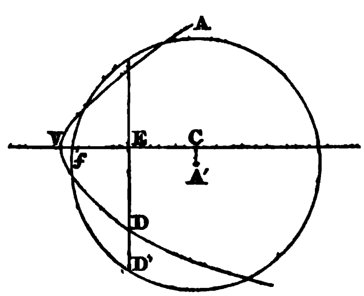


Take any right line passing through A, for the axis of the abscissas x_1, x_2, x_3, x_4, of the points in the right lines R_1S_1, R_2S_2, R_3S_3, R_4S_4, respectively, whose origin is A, and y_1, y_2, y_3, y_4, for the corresponding rectangular co-ordinates of any points in those lines, and X the abscissa, and Y the corresponding rectangular co-ordinate whose origin is A, of any point in the line, to be found; and let the equation of

the respective lines be $x_1=a_1+b_1y_1$, $x_2=a_2+b_2y_2$, $x_3=a_3+b_3y_3$, $x_4=a_4+b_4y_4$, $X=A.Y$; then for the point B_1, let $x_1=X=m_1$, $y_1=Y=n_1$; for B_2 let $x_2=X=m_2$, $y_2=Y=n_2$, for B_3, $x_3=X=m_3$, $y_3=Y=n_3$; and lastly, for B_4, $x_4=X=m_4$, $y_4=Y=n_4$: then we have $m_1=a+bn_1=An_1$; $\therefore n_1=\dfrac{a_1}{A-b_1}$; and similarly we have $n_2=\dfrac{a_2}{A-b_2}$, $n_3=\dfrac{a_3}{A-b_3}$, $n_4=\dfrac{a_4}{A-b_4}$, and also $m_1=An_1$; $\therefore AB_1=n_1.\sqrt{1+A^2}$, and similarly, $AB_2=n_2\sqrt{1+A^2}$, &c.; $\therefore$ by hypothesis, $\dfrac{a_1}{A-b_1}:\dfrac{a_2}{A-b_2}::\dfrac{a_3}{A-b_3}:\dfrac{a_4}{A-b_4}$; that is by reduction $\overline{a_1a_4-a_2a_3}.A^2-(a_1a_4\times\overline{b_1+b_4}-a_2a_3\times\overline{b_2+b_3})A+a_1a_4b_2b_3-a_2a_3b_1b_4=0$; and to elicit a porism from this, I observe, that if $a_1a_4=a_2a_3.a_1a_4.\overline{b_1+b_4}=a_2a_3.\overline{b_2+b_3}$, and $a_1.a_4.b_2.b_3=a_2.a_3.b_1.b_4$, the equation will be identical, and therefore leave A arbitrary. In virtue of the first of these conditions, if neither a_1 nor $a_4=0$, we have from the other two, $b_1+b_4=b_2+b_3$ and $b_2b_3=b_1b_4$; $\therefore (b_1+b_4)^2-4b_1b_4=(b_2+b_3)^2-4b_2b_3$; that is $(b_1-b_4)^2=(b_2-b_3)^2$; $\therefore$ either $b_1-b_4=b_2-b_3$, or $b_1-b_4=b_3-b_2$; the first of these, coupled with the equation, gives $b_1=b_2$ and $b_3=b_4$; and the other, coupled with the equation $b_1+b_4=b_2+b_3$, gives $b_1=b_3$, $b_2=b_4$, that is, the first requires that the two lines S_1R_1 and R_2S_2 are parallel, and also that S_3R_3 and S_4R_4 are parallel; and if the second conditions are to be those adopted, the requisite is that R_1S_1 and R_3S_3 are parallel, and also that R_2S_2, R_4S_4 are parallel; and therefore provided, of the four lines R_1S_1, R_2S_2, R_3S_3, R_4S_4, which are given in position, either R_1S_1, R_2S_2, be parallel, and also R_3S_3, R_4S_4, be parallel, in whatever position they are otherwise given, or else that R_1S_1, R_3S_3, be parallel, and R_2S_2, R_4S_4, be parallel, in whatever position they are in other respects drawn, and in whatever points R_1S_1, R_2S_2, R_3S_3, R_4S_4 cut a line given in position passing through A, provided only that if R_1S_1, R_2S_2, R_3S_3, R_4S_4 cut the given line in D_1, D_2, D_3, D_4, so that $AD_1:AD_2::AD_3:AD_4$, then

any line passing through A, cutting them in points which we will call B_1, B_2, B_3, B_4, will make $AB_1 : AB_2 :: AB_3 : AB_4$, and hence this porism: if a point A be given in position, then in the same plane with it four right lines, R_1S_1, R_2S_2, RS, RS, may be found subject only to those conditions to be found, so that every line drawn through the given point, cutting them respectively in B_1, B_2, B_3, B_4, shall make $AB_1 : AB_2 :: AB_3 : AB_4$.

But there are, in conformity with my observations, other porisms to be derived from the equation $\overline{a_1a_4 - a_2a_3} \, . \, A^2 - \overline{a_1a_4 \, . \, \overline{b_1 + b_4} - a_2a_3 \, . \, \overline{b_2 + b_3}} \, . \, A +$ $a_1a_4b_2b_3 - a_2a_3 \, . \, b_1b_4 = 0$. If the term $a_1a_4b_2b_3 - a_2a_3b_1b_4$ only were equal to 0, A would be given, provided $\dfrac{a_2a_3 \, . \, b_2 + b_4 - a_1a_4 \, . \, b_1 - b_4}{a_1a_4 - a_2a_3}$ were given. If only $a_1a_4 - a_2a_3$ were equal to 0, A would be given, provided $\dfrac{a_2a_3b_2b_3 - a_1a_4 \, . \, b_1 \, . \, b_4}{a_1a_4b_2b_3 - a_2a_3b_1b_4}$ were given in each case, so as to make $AB_1 : AB_2 ::$ $AB_3 : AB_4$. In each case there are only two conditions among eight quantities.



31. I will now add another problem, by way of illustration, to the point. A circle and a parabola, cutting each other, being given in position and magnitude, to find the points of intersection. Let $ax = y^2$ be the equation to the parabola, $\overline{m - x}|^2 +$ $(Y - n)^2 = r^2$ the equation to the circle to the same abscissa x; $\therefore$ for a common point corresponding to the same co-ordinate y, we have $y = Y$, $\therefore$ expunging x we have $\dfrac{y^4}{a^2} - \dfrac{2my^2}{a^2} + m^2 +$ $y^2 - 2ny + n^2 = r^2$, and therefore we have these general propositions allied to porisms. If a parabola cut a circle in four points, the parabola being given in position and magnitude, and from the four points of intersection, lines be drawn which are perpendicular to a line given parallel to

the axis of the parabola, then whatever the radius of that circle may be, provided the centre be given, the sum of the perpendiculars is given, the sum of the rectangles of the squares, and the sum of the solids are all given of those lines intercepted between the points of intersection and the said given line; but if this line had not been said to be given, and parallel to the axis of the parabola, but that only a point through which it passes had been said to be given, these general propositions might have been stated as a porism, or as many porisms, by saying that if a parabola be given in magnitude and position, the centre of a circle given, and a point given, the line passing through that point is given, so that the sum of the perpendiculars is given and required to be shown, the sum of the squares or of their rectangles is given, and required to be shown; and the sum of the solids given and required to be shown, and an infinite number of other functions of those lines, though not, for instance, the sum of fourth powers, &c., will be given, whatever the radius of the circle may be, if it cut the parabola in four points, the truth of this will be evident by the theory of equations, by putting $z = y + e$ in the above equation, so that it may contain all the terms of an equation of the fourth degree, because the radius of the circle is only contained in the last term.

32. It has been well observed, that some of the most common theorems may be so proposed for proof, that the proposition might be a real porism. For instance, we might say, an ellipsis being given, two points are given in it, so that the sum of the right lines drawn from those points to any point in the circumference is given.

SCHOLIUM, NOT INCLUDED IN THIS LETTER.

Porismatic expressions and porisms may be transformed into other porismatic expression and other porisms: for instance, take the porismatic expression—

$$\sqrt[2]{2 - \overline{2 - \sqrt[2]{2}.\rho + \sqrt[2]{2 - 4\rho}}} + \sqrt[2]{2 - \overline{2 - \sqrt[2]{2}.\rho - \sqrt[2]{2 - 4\rho}}} = \sqrt[2]{4 + 2\sqrt[2]{2}};$$

and put $\sqrt[2]{2 - 4\rho} = 2.(1 - \rho')$; $\therefore 2 - 4\rho = 4 - 8\rho' + 4\rho'^2$; $\therefore \rho = -\frac{1}{2} + 2\rho' - \rho'^2$ and consequently

$$\sqrt[2]{5 - \tfrac{1}{2}\sqrt[2]{2} - \overline{6 - 2\sqrt[2]{2}.\rho' + 2 - \sqrt[2]{2}.\rho'^2}} + \sqrt[2]{1 - \tfrac{1}{2}\sqrt[2]{2} - \overline{2 - 2\sqrt[2]{2}.\rho' + 2 - \sqrt[2]{2}.\rho'^2}} = \sqrt[2]{4 + 2\sqrt[2]{2}},$$

whatever ρ' may be. But, I remark, which appears necessary, in consequence of its general importance, that such expressions, having more than one root, the equality, as I have before remarked, does not refer to them all; but, also, that though the expression may, according to its meaning, be universally of the nature stated, whatever may be the value of the arbitrary quantities, and though the equation be of the nature that all such arbitrary quantities will vanish when any attempt is made to discover their value, still the equality in such equations as the above does not exist universally with the same order of roots; thus, in the above expression, though we may allow the positive sign to be intended to be used on the right-hand side of the equation, to the marks $\sqrt[2]{\ }$, still, with some values of ρ, the positive sign, and with other values of ρ, the negative sign, must be used with the sign $\sqrt[2]{\ }$, prefixed externally to the two expressions on the left-hand side, and this is in complete accordance with, and a good example to, my idea of partial equality used in my Tracts on Imaginary Quantities.' And though it may not always be necessary to

introduce a particular sign to denote a partial equality, as it may be sufficient to leave that to the reader's judgment, it may be beneficial, and may even avoid error, to use a distinguishing sign, and for the present occasion I suggest the sign $\underset{-}{\overset{+}{?}}$, that is, the note of interrogation between the two signs + and −. Thus, alluding to the equation above, and writing it thus,

$$\underset{-}{\overset{+}{?}}\sqrt{2-\overline{2-\sqrt[2]{2}}\cdot\rho+\sqrt[2]{2-4\rho}}\ \underset{-}{\overset{+}{?}}\ \sqrt[2]{2-\overline{2-\sqrt[2]{2}}\cdot\rho-\sqrt[2]{2-4\rho}}=\sqrt{4+2\sqrt{2}},$$

we should mean, that whatever ρ might be, the left-hand side of the equation will be equal to the right-hand side, with the understanding that the sign $\underset{-}{\overset{+}{?}}$ signifies a doubt whether it is to be taken + or −; but which of the two it should be, depending on the value of ρ; this equation, if we do, or do not use the doubtful sign, but leave the doubt as in common cases to be considered, to be expressed by the sign itself $\sqrt[2]{\ }$, will, as the reader will no doubt recollect, have no criterion for determining ρ, as it will vanish from the equation by a proper development. But there is a means in the above equation of finding what values of ρ will require either term on the left-hand side, to have the sign $\underset{-}{\overset{+}{?}}$, interpreted by + or by −, with respect to the equation

$$\sqrt[2]{2-\overline{2-\sqrt[2]{2}}\cdot\rho+\sqrt[2]{2-4\rho}}+\sqrt{2-\overline{2-\sqrt[2]{2}}\cdot\rho-\sqrt[2]{2-4\rho}}.$$

Thus; considering that the first $\underset{-}{\overset{+}{?}}$ is to be taken as positive, all values of ρ which make $2-\overline{2-2\sqrt[2]{\rho}}+\sqrt[2]{2-4\rho}$ less than $4+2\sqrt[2]{2}$ require the second $\underset{-}{\overset{+}{?}}$ to be positive; and the contrary.

The above equations are not equations to a porismatic problem, though they are of a porismatic character; but the following is a porism which will illustrate the importance of the above remark: x and y are given so that $\sqrt[2]{x-6-\overline{2\sqrt[2]{2}}\cdot\rho+y\cdot\rho^2}+\sqrt[2]{\tfrac{1}{2}y-\overline{2y-2}\cdot\rho+y\rho^2}$, shall be given, whatever ρ may be. I express this in the common mode, without alluding

to the circumstance that, in reality, the sign $\sqrt[2]{}$ in each part, for the porism to be true, should be, (unless it be intended that the two parts be equal to each other, in which case, they must have contrary signs, and the sum $= 0$,) prefixed by the sign $\underline{\stackrel{+}{?}}$. To prove this porism, put the expression $= p$, which is to be shown to be given as well as x and y, and we shall have $\sqrt[2]{x - \overline{6 - 2\sqrt[2]{2}}\rho + y\rho^2} = p - \sqrt[2]{\tfrac{1}{2}y - \overline{2y - 2}.\rho + y\rho^2}$; square, and we get $x - \overline{6 - \sqrt[2]{2}}.\rho + y\rho^2 = p^2 - 2p.\sqrt[2]{\tfrac{1}{2}y - \overline{2y - 2}.\rho + y\rho^2} + \tfrac{1}{2}y - \overline{2y - 2\sqrt[2]{2}}.\rho + y\rho^2$; put for the sake of brevity,

$A = \dfrac{x - p^2 - \frac{1}{2}y}{2p}$, and $B = \dfrac{8 - 2\sqrt[2]{2} - 2y}{2p}$; $B\rho - A = \sqrt{\tfrac{1}{2}y - \overline{2y - 2}.\rho + y\rho^2}$,*

consequently, by squaring, since ρ is to disappear, we find

$A^2 = \tfrac{1}{2}y$, $A.B = y - 1$, $B^2 = y$ $\therefore$ $(A^2B^2 =)\, y^2 - 2y + 1 = \tfrac{1}{2}y^2$, $y^2 - 4y + 2 = 0$; $y = 2 \pm \sqrt{2}$, $p = \dfrac{4 - \sqrt[2]{2} - y}{B}$; $x = p^2 + 2Ap + \tfrac{1}{2}y = (p + A)^2$; because $\tfrac{1}{2}y = A^2$; or because $p + A = \dfrac{4 - \sqrt[2]{2} - y}{B} + A = \dfrac{4 - \sqrt[2]{2} - y + A.B}{B} = \dfrac{3 - \sqrt[2]{2}}{B}$; because $A.B - y = -1$; consequently $x = \left(\dfrac{3 - \sqrt[2]{2}}{B}\right) = \dfrac{11 - 6\sqrt[2]{2}}{2} \times (2 \mp \sqrt[2]{2})$; by taking $y = 2 \pm \sqrt{2}$, or which is the same thing, by taking $B = \sqrt{2 \pm \sqrt{2}}$ and, by a little reduction, we therefore get either $x = 17 - 23\sqrt{\tfrac{1}{2}}$, $p = 2\sqrt[2]{5 - 7\sqrt[2]{\tfrac{1}{2}}}$, y being $= 2 + \sqrt[2]{2}$ or; $x = 5 - \tfrac{1}{2}\sqrt{2}$, $p = \sqrt{4 + 2\sqrt[2]{2}}$, y being $= 2 - \sqrt[2]{2}$; observing that $\sqrt[2]{2}$ and $\sqrt[2]{\tfrac{1}{2}}$ are meant to express positive quantities, and for which purpose it ought to be marked in conformity with strict notation in a particular way, which might, for instance, if so agreed on, be marked thus $\underset{\cdot}{\sqrt[2]{2}}$ and $\underset{\cdot}{\sqrt[2]{\tfrac{1}{2}}}$; some such notation being required

* I exclude the case of $p = o$ as not being within the meaning of the porism, and which would require the two portions $\sqrt[2]{x - \overline{6 - 2\sqrt[2]{2}}.\rho + y\rho^2}$ and $\sqrt[2]{\tfrac{1}{2}y - \overline{2y - 2}.\rho + y\rho^2}$ to be equal and of contrary signs, and for which purpose it would be required that $\overline{2y - 2} = \overline{6 - \sqrt[2]{2}}$, or $y = 4 - \sqrt{2}$ and $x = \tfrac{1}{2}y$.

as the sign $\sqrt[2]{\ }$, is of a double character; from the above investigation, we see that x and y are given, so that

$$\overset{+}{\underset{-}{?}}\sqrt{x-\overline{6-2\sqrt[2]{2}}\,.\,\rho+y\,.\,\rho^2}\ \overset{+}{\underset{-}{?}}\sqrt{\tfrac{1}{2}y-2\,.\,\overline{y-2}\,.\,\rho+y\,.\,\rho^2},$$

shall be given, and that we have two solutions to the problem.

But I think it proper to call the reader's attention, as it may be required in the investigations of the truth of a porism very frequently, whether the subject be of a pure geometrical nature, or of a symbolic nature, when the investigation is undertaken by means of cases of ease, and the proposition is not expressed by means of such characters as $\overset{\times}{\underset{-}{?}}$; that we may hit on a case of ease which, without due attention, would lead us wrong, and cause us to conclude that the ~~association~~ assertion of the truth of the porism was false. Thus, suppose we asserted, in commonly used symbols, that x and y were given, so that $\sqrt{x-\overline{6-2\sqrt[2]{2}}\,.\,\rho+y\rho^2}+\sqrt{\tfrac{1}{2}y-\overline{2y-2}\,.\,\rho+y\rho^2}$ should be given; and if to investigate the truth of this, we put it $=p$, and assumed two cases of ease, the one that ρ were $=0$, the other that $y\,.\,\rho^2-\overline{6-2\sqrt[2]{2}}\,.\,\rho=0$; ($\rho$ not being supposed $=0$;) which last assumption would cause ρ to be equal to $\frac{6-2\sqrt[2]{2}}{y}$, and we should get from the two expressions the following two equations, $\sqrt[2]{x}+\sqrt[2]{\tfrac{1}{2}y}=p$, and $\sqrt[2]{x}+\sqrt[2]{\tfrac{1}{2}y-\overline{2y-2-6+2\sqrt[2]{2}}\times\frac{6-2\sqrt[2]{2}}{y}}=p$. These two equations, if supposed true, without restriction, would give $y=4-\sqrt[2]{2}$; which would not be true, and would lead to the presumption that the porisms did not exist. And the reason of this is, that one assumption of ρ, as we above stated might occur, which might not allow the same signs to be taken before the character $\surd$, that the other assumption of ρ might require: thus, if we take two separate values of ρ, the one 0, and the other very great and indefinitely so, and we do not absolutely use the character $\overset{+}{\underset{-}{?}}$, but write the two cases, $\sqrt{x}+\sqrt{\tfrac{1}{2}y}=p$, and

$$\sqrt{x - \overline{6 - 2\sqrt[2]{2}} . \rho + y\rho^2} + \sqrt{\tfrac{1}{2}y - \overline{2y - 2} . \rho + y\rho^2} = p,$$

and observe, with regard to the last equation, that if ρ be indefinitely large, that the left-hand side will be indefinitely near

$$\sqrt{x} \times \left(\rho - \frac{6 - 2\sqrt[2]{2}}{2x}\right) + \sqrt{\frac{y}{2}} \times \left(\rho - \frac{2y - 2}{y}\right) = p,$$

it is evident this cannot be true whilst ρ is indeterminate, unless $\sqrt{y}$ in the first portion has a contrary sign affixed to it, that it has in the second portion of the expression, consequently if ρ is large beyond a certain limit, if the porism be true, the two terms multiplied by $\sqrt{y}$ must have contrary signs, consequently under that restriction we have $\dfrac{6 - 2\sqrt[2]{2}}{2\sqrt{x}} - \dfrac{2y - 2}{2\sqrt{y}} = p$;

that is, $p = \dfrac{4 - \sqrt[2]{2} - y}{\sqrt{y}}$, $\sqrt{x} = p - \sqrt[2]{\tfrac{1}{2}y} = \dfrac{4 - \sqrt[2]{2} - \overline{1 + \sqrt[2]{\tfrac{1}{2}}}y}{2\sqrt[2]{y}}$; but still we require another equation, as there are three quantities p, x, y to be determined. For this purpose, take ρ extremely small, but indefinitely so, and not $= 0$, then the equation, by omitting ρ^2 becomes

$\sqrt{x - \overline{6 - 2\sqrt[2]{2}} . \rho} + \sqrt{\tfrac{1}{2}y - \overline{2y - 2} . \rho} = p$, that is, by extraction, omitting ρ^2, &c.,

$\sqrt{x} \times \left(\rho - \dfrac{6 - 2\sqrt[2]{2}}{2x}\right) + \sqrt{\dfrac{y}{2}} \times \left(\rho - \dfrac{2y - 2}{y}\right) = p$; from this take the equation

$\sqrt{x} + \sqrt{\tfrac{1}{2}y} = p$; divide by ρ, and transpose, and we obtain

$\sqrt{x} = \dfrac{(3 - \sqrt[2]{2}) . \sqrt[2]{y}}{\sqrt{2} . (1 - y)}$; but we have had also $\sqrt{x} = \dfrac{4 - \sqrt[2]{2} - (1 + \sqrt[2]{\tfrac{1}{2}})y}{\sqrt{y}}$; put these two values equal to each other; and, by a little reduction, we get $y^2 - \overline{6 - 2\sqrt{2}} . y + 10 - 6\sqrt{2} = 0$; and we get $y = 3 - \sqrt{2} \pm 1$, that is $y = 2 - \sqrt{2}$, and $4 - \sqrt{2}$; and I observe that $\sqrt{}$ admits of the double value, corresponding only to the double value of $\sqrt{2}$ in the formula, so that, though we had $y = 2 - \sqrt{2}$, and $y = 2 + \sqrt{2}$; which of the two it should be, depends on which way $\sqrt{2}$ in the formula is taken, so that, in fact, when that is determined, the porism is only true with one of the values. I exclude the case of $y = 4 - \sqrt{2}$, as stated in the note above, since that makes $p = 0$, &c.

By way of amusing the reader,

> "ut pueris olim dant crustula blandi
> Doctores,"

I will offer for his reflection the following equations, which he may, if inclined to use a jocular definition, call Equations of Ghosts, or, if he pleases, Arbitrary Spirits; as the values of some of the letters in the equation are quite arbitrary, without destroying the truth of the equation, and are so assumed, that if any proper attempt be made to discover their value, by any method of resolving equations, they will vanish with all the terms known or sought in the equation:

1st. Let it be proposed to determine ρ, if possible, in the equation

$$\sqrt[2]{5-\tfrac{1}{2}\sqrt[2]{2}-\overline{6-2\sqrt[2]{2}}.\rho+\overline{2-\sqrt[2]{2}}.\rho^2}+\sqrt{1-\tfrac{1}{2}\sqrt{2}-\overline{2-2\sqrt[2]{2}}.\rho+\overline{2-\sqrt[2]{2}}.\rho^2}=\sqrt{4+2\sqrt[2]{2}}.$$

2d. Let it be required to find ρ and ρ', if possible, in the equation

$$\sqrt[2]{5-\tfrac{1}{2}\sqrt[2]{2}-\overline{6-2\sqrt[2]{2}}.\rho+\overline{2-\sqrt[2]{2}}.\rho^2}+\sqrt[2]{1-\tfrac{1}{2}\sqrt[2]{2}-\overline{2-2\sqrt[2]{2}}.\rho+\overline{2-\sqrt[2]{2}}.\rho^2}=$$
$$\sqrt[2]{5-\tfrac{1}{2}\sqrt[2]{2}-\overline{12-4\sqrt[2]{2}}.\rho'+\overline{8-4\sqrt[2]{2}}.\rho'^2}+\sqrt[2]{1-\tfrac{1}{2}\sqrt[2]{2}-\overline{4-4\sqrt[2]{2}}.\rho'-\overline{8-4\sqrt[2]{2}}.\rho'^2}.$$

3d. Let it be required to find ρ, ρ', ρ'', and x, from the following equations.

$$\sqrt[2]{5-\tfrac{1}{2}x-\overline{6-2}.x+\overline{2-x}.\rho^2}+\sqrt[2]{1-\tfrac{1}{2}x-\overline{2-2x}.\rho+\overline{2-x}.\rho^2}=\sqrt[2]{5-\tfrac{1}{2}x-\overline{12-4x}.\rho'+\overline{8-4x}.\rho'^2}+$$
$$\sqrt[2]{1-\tfrac{1}{2}x-\overline{4-4x}.\rho'+\overline{8-4x}.\rho'^2};\ \text{and } x=\sqrt[3]{\frac{3+\rho''}{\sqrt{2}}+\sqrt[2]{\frac{(3+\rho'')^2}{2}+\left(\frac{1+\rho''}{3}\right)^3}}+$$
$$\sqrt[3]{\frac{3+\rho''}{\sqrt{2}}-\sqrt[2]{\frac{(3+\rho'')^2}{2}+\left(\frac{1-\rho}{3}\right)^3}}$$; and it will be found that x will be a real value which will be determinable, and that ρ, ρ', and ρ'', are all arbitrary. I observe that I have not introduced before the sign $\surd$, the mark I have suggested of $\overset{+}{?}$; for that mark of doubtful, or probably, more properly speaking, changeable sign depending on its character on the value of the arbitrary quantities; as whether that sign be introduced or not the arbitrary quantities would vanish when the proper means are used in an attempt to discover their value.

It was my wish to add a few more remarks on the subject of Porisms, including that of "Porismatic Tendencies," and on Partial Equality, &c.; and to add something on subjects connected either directly or indirectly with this discussion, in Astronomy, &c.; but if I find myself capable of again taking up the subject, which I feel an inclination to do, I am compelled to leave it for another occasion.

The reader is reminded, in the event of its being necessary, that the determining of the data by means of cases of ease is not a demonstration of the truth of the porism, but merely a preliminary to it.

ERRATA TO THE IMAGINARY QUANTITIES.

BOOK I.

Page 11, lines 4 and 9, *for* 2cspr, *read* 2csprq.

„ 12, *for* $\frac{\overline{b+\rho x}|^{3}}{2}$, *read* $\overline{\frac{b+\rho x}{2}}\Big|^{3}$; and *for* $\frac{\overline{a-\rho}|^{3}}{3}$, *read* $\overline{\frac{a-\rho}{3}}\Big|^{3}$ everywhere; and *for* $+\frac{\overline{a_{i}-\rho}|^{3}}{3}$ *read* $-\overline{\frac{a-\rho}{3}}\Big|^{3}$; and observe the same in page 14.

„ 16, 8 lines from the top, *for* $=1$, *read* $=3$.

„ 17, 1 line from the top, *for* $\frac{3}{2\rho}$, *read* $\frac{3}{2}\cdot\rho$; 4 lines from top, *for* $-\rho$, *read* $+\rho$ in both places.

„ 18, 8 lines from the top, *for* $\sqrt{-\frac{3}{4}}$, *read* $\sqrt{\rho-\frac{3}{4}}$.

„ 22, 5 lines from the top, *for* hyp. log. of $\frac{\sqrt{-1}}{\sqrt{-1}}$, *read* $\frac{\text{hyp. log. of } \sqrt{-1}}{\sqrt{-1}}$.

„ 23, 7 lines from the top, *for* the 12st R_{n}, *read* R.

„ 26, 15 lines from the top, *for* 76^{8}, *read* 72°.

„ 27, 5 lines from the top, *for* $z=o$, *read* $5z=o$; after line 5, and in the note, *for* Q′, *read* Q″, and add that $Q''=A.\varepsilon^{mx}.Q'$.

„ 28, at the bottom, *for* Δ^{n}, *read* $\Delta^{n}{}_{n}$.

„ 30, lines 2, 3, 4 from the top, *for* $a^{2}-b^{2}$, *read* $a^{2}+b^{2}$.

„ 34, line 16, *for* q evanescent, *read* r evanescent.

„ 35, *for* r in the formulæ *read* ρ.

BOOK II.

Read everywhere Wallis *for* Wallace.

Page viii, in the diagram, *put* F *for* P.

„ x, line 6 from the bottom, *for* DP *read* BP.

„ 12, line 5 from the bottom, *for* a . PQ=), *read* a . PQ)=; last line, *for* AK, *read* AK′.

„ 13, line 3, *for* m *read* n; line 5, *for* nP, Gn, *read* mP . Gm.

„ 14, line 8 from the top, before AG=, place a comma.

„ 10, line 2 from the bottom, before sine, place a comma.

„ 19, line 19 from the bottom, before sine, place a comma.